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Radiant heat transfer between grey surfaces: an alternative approach

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Abstract

The paper proposes an alternative approach for reformulating the problem of heat interchange between grey surfaces. An integral Fredholm equation of second kind is still found, as for the usual treatment, but the unknown function depends only on geometry and surface properties, being independent of local surface temperature and heat flux, which can then be related to each other in a simple way by the solution of the integral equation. Examples of the solution of the integral equation by series or by decomposition in a complete set of orthogonal normalised functions are reported.

1. Introduction

Radiant interchange between non-black surfaces is a fundamental problem encountered in a variety of engineering situations. The general formulation of the problem requires energy balances at any surface location, taking into account energy arriving from all directions in space. A main assumption is usually made in tackling the problem: reflection and emission from participating surfaces is diffused, i.e. Lambert's Law is obeyed. This assumption implies that radiation directionality need not be considered. A further assumption is sometime made to simplify the problem: the incident energy flux on a given surface is uniformly distributed. Such assumption allows the formulation of relatively simple calculation procedures [1] from which radiation quantities (such as mean radiosity or over-all-heat transfer rate) can be found by solving a set of linear algebraic equations.

However, except for the simplest geometry, the incident energy flux is not uniform even for uniform temperature distributions over the surfaces. This implies that local heat flux may vary over the surfaces and that over-all-heat transfer rate evaluation may not be accurate enough.

The solution of the problem after lifting the simplifying assumption of uniform incident energy flux leads to an integral equation for the distribution of the surface heat flux or the surface temperature (see e.g. [2,3]). The governing equation is an integral Fredholm equation of the second kind and its general form is:

$$\phi(\mathbf{x}) = f(\mathbf{x}) + \int_A K(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})\,\mathrm{d}A_y \tag{1}$$

where $f(\mathbf{x})$ is a known function (e.g. a function of the surface temperature) and $\phi(\mathbf{x})$ is the unknown to be determined (e.g. radiosity or local heat flux). A part of very simple situations (e.g. spherical cavity with uniform surface properties), analytical solutions of such equation do not exist, whereas numerical solutions can be found in a number of ways [3]: (i) successive approximations; (ii) reduction to algebraic equation by

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Nomenclature

a, b, c	linear dimensions (m)
A_j	area of the jth surface (m^2)
$B(\mathbf{x}, \mathbf{y})$	function (m^{-2})
G	irradiance (W m^{-2})
$H(\mathbf{x}, \mathbf{y})$	resolvent kernel of integral equation
J	radiosity (W m^{-2})
$K(\mathbf{x}, \mathbf{y})$	kernel of integral equation
R	distance (m)
R_0	sphere radius (m)
T	temperature (K)
$\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$	vector position
$Z_n(\mathbf{x})$	normalised Legendre polynomial
α, β	coefficient
$\delta(\mathbf{x})$	Dirac delta function
δ_{mn}	Kroeneker symbol
ε	emittance
φ	thermal flux (W m^{-2})
$\varphi_n^{(k)}(\mathbf{x})$	complete set of orthogonal normalised functions defined on kth surface
θ	angle (rad)
ρ	reflectivity
σ	Stefan–Boltzman constant (W m^{-2} K^{-4})
ξ, η, ζ	co-ordinates on plane surfaces

numerical quadrature; (iii) Taylor series expansion [4]; (iv) approximate separable kernels [5]; (v) variational methods [6,7]; (vi) transformation in integro-differential equation (Ambarzumian method) [8]. The alternative approach proposed here leads again to an integral equation where the unknown function depends only on geometry and surface properties (like reflectivity) but not on temperature or heat flux. The knowledge of that function would allow to solve any radiant interchange problem relative to that geometry and surface properties. Moreover, in the quite common case where the cavity can be decomposed into a finite number of surface having uniform properties, the unknown function can be written as a converging series of functions depending only on the geometry (that can then be evaluated once for ever) and only the coefficients of the series depend in a simple way on the surface properties. Finally, for those situations where the series converges too slowly, an alternative method can be applied, based on the use of complete ortho-normal sets of functions, an example of application is given.

2. The governing equation

Let us consider two surfaces (i, k) of radiosity $J_i(\mathbf{x}_i)$ and $J_k(\mathbf{x}_k)$ where $\mathbf{x}$ is the point co-ordinate on the surfaces. The power arriving onto k from i can be evaluated (for diffuse surfaces) as:

$$\delta Q_{i\to k} = J_i \frac{\cos(\theta_i)\cos(\theta_k)}{\pi R^2} \mathrm{d}A_i\, \mathrm{d}A_k$$

referring to Fig. 1 for symbols.

As θ_i, θ_k and R are both function of $\mathbf{x}_i$ and $\mathbf{x}_k$ the group $a(\mathbf{x}_i, \mathbf{x}_k) = \frac{\cos(\theta_i)\cos(\theta_k)}{\pi R^2}$ is a function of $(\mathbf{x}_i, \mathbf{x}_k)$. The irradiation arriving on $\mathbf{x}_k$ can then be evaluated as:

$$G(\mathbf{x}_k) = \int_A J_i(\mathbf{x}_i) a(\mathbf{x}_i, \mathbf{x}_k)\, \mathrm{d}A_i \tag{2}$$

where the integration should be extended to all the surfaces forming a cavity.

Let us now consider a grey cavity, the temperature and emittance of the surface will be considered a function of position $\mathbf{x}$. The energy balance at each point of the surface gives:

$$\varphi(\mathbf{x}) = J(\mathbf{x}) - G(\mathbf{x}) \tag{3}$$

where $\varphi(\mathbf{x})$ is the heat flux, $J(\mathbf{x}) = \varepsilon(\mathbf{x})\sigma T^4(\mathbf{x}) + \rho(\mathbf{x})G(\mathbf{x})$ is the radiosity and $\rho(\mathbf{x}) = 1 - \alpha(\mathbf{x}) = 1 - \varepsilon(\mathbf{x})$.

The usual formulation of the problem is obtained by eliminating $\varphi(\mathbf{x})$ and $G(\mathbf{x})$ from Eqs. (2) and (3) and the definition of radiosity, yielding:

$$J(\mathbf{x}) = \varepsilon(\mathbf{x})\sigma T^4(\mathbf{x}) + (1 - \varepsilon(\mathbf{x})) \int_A J(\mathbf{y}) a(\mathbf{y}, \mathbf{x})\, \mathrm{d}A_y \tag{4}$$

Let us now consider the following re-formulation of the problem: Eq. (4) can be rewritten as:

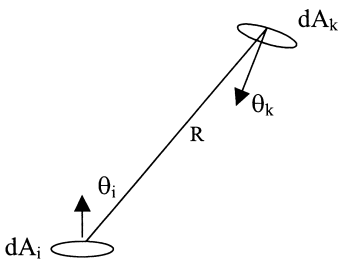

Fig. 1. Couple of infinitesimal surfaces exchanging energy by radiation.

$$\int_A J(\mathbf{y})\big[\delta(\mathbf{y}-\mathbf{x})-(1-\varepsilon(\mathbf{x}))a(\mathbf{y},\mathbf{x})\big]\,\mathrm{d}A_y=\varepsilon(\mathbf{x})\sigma T^4(\mathbf{x}) \tag{5}$$

where $\delta(\mathbf{y}-\mathbf{x})$ is the Dirac delta function.

Defining: $B(\mathbf{y},\mathbf{x})=\delta(\mathbf{y}-\mathbf{x})-(1-\varepsilon(\mathbf{x}))a(\mathbf{y},\mathbf{x})$, Eq. (5) can be solved if a function $B^{-1}(\mathbf{x},\mathbf{z})$ exits such that:

$$\int_A B(\mathbf{y},\mathbf{x})B^{-1}(\mathbf{x},\mathbf{z})\,\mathrm{d}A_x=\delta(\mathbf{y}-\mathbf{z}) \tag{6}$$

in fact, multiplying Eq. (5) by $B^{-1}(\mathbf{x},\mathbf{z})$ and integrating yields:

$$J(\mathbf{z})=\int_A \varepsilon(\mathbf{x})\sigma T^4(\mathbf{x})B^{-1}(\mathbf{x},\mathbf{z})\,\mathrm{d}A_x \tag{7}$$

A function $b(\mathbf{x},\mathbf{z})$ can be defined as:

$$(1-\varepsilon(\mathbf{z}))b(\mathbf{x},\mathbf{z})=\delta(\mathbf{x}-\mathbf{z})-B^{-1}(\mathbf{x},\mathbf{z}) \tag{8}$$

and substitution into Eq. (6) gives:

$$\begin{aligned}\delta(\mathbf{y}-\mathbf{z})=&\int_A\big[\delta(\mathbf{y}-\mathbf{x})-(1-\varepsilon(\mathbf{x}))a(\mathbf{y},\mathbf{x})\big]\big[\delta(\mathbf{x}-\mathbf{z})\\&+(1-\varepsilon(\mathbf{z}))b(\mathbf{x},\mathbf{z})\big]\,\mathrm{d}A_x=\delta(\mathbf{y}-\mathbf{z})\\&-(1-\varepsilon(\mathbf{z}))a(\mathbf{y},\mathbf{z})+(1-\varepsilon(\mathbf{z}))b(\mathbf{y},\mathbf{z})\\&-(1-\varepsilon(\mathbf{z}))\int_A(1-\varepsilon(\mathbf{x}))a(\mathbf{y},\mathbf{x})b(\mathbf{x},\mathbf{z})\,\mathrm{d}A_x\end{aligned}$$

or:

$$b(\mathbf{y},\mathbf{z})-a(\mathbf{y},\mathbf{z})=\int_A(1-\varepsilon(\mathbf{x}))a(\mathbf{y},\mathbf{x})b(\mathbf{x},\mathbf{z})\,\mathrm{d}A_x \tag{9}$$

The integral Eq. (9) gives the solution of the problem: after finding $b(\mathbf{y},\mathbf{z})$, the function $B^{-1}(\mathbf{y},\mathbf{z})$ can be computed by Eq. (8) and Eq. (7) becomes:

$$\begin{aligned}J(\mathbf{z})=&\varepsilon(\mathbf{z})\sigma T^4(\mathbf{z})\\&+(1-\varepsilon(\mathbf{z}))\int_{A_x}\varepsilon(\boldsymbol{x})\sigma T^4(\mathbf{x})b(\mathbf{x},\mathbf{z})\,\mathrm{d}A_x\end{aligned} \tag{10}$$

Further substitution into Eqs. (2) and (3) yields:

$$G(\mathbf{x})=\int_A\varepsilon(\mathbf{z})\sigma T^4(\mathbf{z})b(\mathbf{z},\mathbf{x})\,\mathrm{d}A_z \tag{11}$$

$$\varphi(\mathbf{x})=\varepsilon(\mathbf{x})\left[\sigma T^4(\mathbf{x})-\int_A\varepsilon(\mathbf{z})\sigma T^4(\mathbf{z})b(\mathbf{z},\mathbf{x})\,\mathrm{d}A_z\right] \tag{12}$$

The function $b(\mathbf{x},\mathbf{z})$ depends only on the surface geometry (through $a(\mathbf{x},\mathbf{y})$) and properties (through $\varepsilon(\mathbf{x})$) and Eqs. (11) and (12) relate surface temperature to local heat flux once the function $b(\mathbf{z},\mathbf{x})$, which is independent of $T(\mathbf{x})$ and $\varphi(\mathbf{x})$, is known. This particular reformulation of the problem allows to separate the geometry and properties of the cavity from the particular temperature and heat flux field.

Eq. (9) is again a Fredholm integral equation of second kind, as Eq. (1), but once the solution is found for a given cavity, the problem of finding the local heat flux from the temperature field (or viceversa) is solved for any temperature field (or heat flux field).

An advantage of this reformulation can be recognised in unsteady problems: when heat flux and surface temperature fields are varying with time, Eq. (12) can always be used to relate them as the function $b(\mathbf{z},\mathbf{x})$ does not change with time (if surface properties are not changing too). Then the integral equation must be solved once and the solution used to compute $\varphi(\mathbf{x})$ or $T(\mathbf{x})$ at each time step.

3. The solution of the integral equation

The integral equation (9) can be rewritten in a more symmetric form as:

$$b(\mathbf{y},\mathbf{z})=a(\mathbf{y},\mathbf{z})+\int_A\rho(\mathbf{x})a(\mathbf{y},\mathbf{x})b(\mathbf{x},\mathbf{z})\,\mathrm{d}A_x \tag{9b}$$

whose solution can be written as:

$$b(\mathbf{x},\mathbf{z})=a(\mathbf{x},\mathbf{z})+\int_A H(\mathbf{x},\mathbf{y})a(\mathbf{y},\mathbf{z})\,\mathrm{d}A_y$$

where $H(\mathbf{x},\mathbf{y})=\rho(\mathbf{y})H_0(\mathbf{x},\mathbf{y})$ is the resolvent kernel (see e.g. [9,10]) and:

$$H_0(\mathbf{x},\mathbf{y})=a(\mathbf{x},\mathbf{y})+\sum_{n=2}^{\infty}H_n(\mathbf{x},\mathbf{y})$$

where:

$$H_2(\mathbf{x},\mathbf{y})=\int_A a(\mathbf{x},\mathbf{z})\rho(\mathbf{z})a(\mathbf{z},\mathbf{y})\,\mathrm{d}A_z,$$

and generally:

$$H_{n+1}(\mathbf{x},\mathbf{y})=\int_A a(\mathbf{x},\mathbf{z})\rho(\mathbf{z})H_n(\mathbf{z},\mathbf{y})\,\mathrm{d}A_z$$

The proof can be found in Appendix A.

Only in very few cases, the functions $H_n(\mathbf{x},\mathbf{y})$ can be calculated exactly, depending on the complexity of the functions $a(\mathbf{x},\mathbf{y})$ and $\rho(\mathbf{y})$.

A typical situation is that where the cavity surfaces can be decomposed into N surfaces having uniform radiative properties (i.e. constant values of $\rho(\mathbf{x})$).

In this case, Eq. (9b) can be rewritten as:

$$b(\mathbf{x},\mathbf{y})=a(\mathbf{x},\mathbf{y})+\sum_{k=1}^{N}\rho_k\int_{A_k}a(\mathbf{x},\mathbf{z}_k)b(\mathbf{z}_k,\mathbf{y})\,\mathrm{d}A_{z_k} \tag{9c}$$

and its solution is:

$$b(\mathbf{x},\mathbf{y}) = a(\mathbf{x},\mathbf{y}) = \sum_{k=1}^{N}\rho_k \int_{A_k} H_0(\mathbf{x},\mathbf{z}_k)a(\mathbf{z}_k,\mathbf{y})\,\mathrm{d}A_k$$

or:

$$b(\mathbf{x},\mathbf{y}) = a(\mathbf{x},\mathbf{y}) + \sum_{k=1}^{N}\rho_k Y^k_{(2)}(\mathbf{x},\mathbf{y}) + \sum_{k,\,l=1}^{N}\rho_k\rho_l Y^{kl}_{(3)}(\mathbf{x},\mathbf{y}) + \cdots \tag{13}$$

where:

$$Y^k_{(2)}(\mathbf{x},\mathbf{y}) = \int_{A_k} a(\mathbf{x},\mathbf{z}_k)a(\mathbf{z}_k,\mathbf{y})\,\mathrm{d}A_{z_k}$$

$$Y^{kl}_{(3)}(\mathbf{x},\mathbf{y}) = \int_{A_k}\int_{A_l} a(\mathbf{x},\mathbf{z}_k)a(\mathbf{z}_k,\mathbf{z}_l)a(\mathbf{z}_l,\mathbf{y})\,\mathrm{d}A_{z_k}\mathrm{d}A_{z_l}$$

$$\vdots$$

and now the functions $Y_{(a)}(\mathbf{x},\mathbf{y})$ depend only on the cavity geometry. The independent variables in $a(\mathbf{x},\mathbf{y})$, $b(\mathbf{x},\mathbf{y})$ $Y(\mathbf{x},\mathbf{y})$ span over all the cavity surfaces, but it is possible to split each function in a finite number $(N\times N)$ of functions whose independent variables span over a couple of surfaces: $a_{ik}(\mathbf{x}_i,\ \mathbf{y}_k)$, $b_{ik}(\mathbf{x}_i,\ \mathbf{y}_k)$, $Y^m_{2,\,ik}(\mathbf{x}_i,\ \mathbf{y}_k)$, etc. The series Eq. (13) converges rapidly if the reflectivities ρ_i are small; for example, if the cavity is made by two half-spheres of radius R_0 with equal and uniform reflectivity ρ:

$$a_{ik}(\mathbf{x}_i,\mathbf{x}_k) = \frac{1}{4\pi R_0^2} \quad i,k = 1,2$$

and:

$$Y^m_{(2)i,\,k}(\mathbf{x}_i,\mathbf{y}_k) = \int_{A_m} a(\mathbf{x}_i,\mathbf{z}_m)a(\mathbf{z}_m,\mathbf{y}_k)\,\mathrm{d}A_{z_m} = \frac{1}{\left(4\pi R_0^2\right)^2}\int_{A_m}\mathrm{d}A_{z_m} = \frac{A_m}{\left(4\pi R_0^2\right)^2} = \frac{1}{8\pi R_0^2}$$

$$Y^{m,\,n}_{(3)i,\,k} = \int_{A_m}\int_{A_n} a(\mathbf{x}_i,\mathbf{z}_m)a(\mathbf{z}_m,\mathbf{z}_n)a(\mathbf{z}_n,\mathbf{y}_k)\,\mathrm{d}A_{z_m}\,\mathrm{d}A_{z_n} = \frac{1}{\left(4\pi R_0^2\right)^3}\int_{A_m}\int_{A_n}\mathrm{d}A_{z_m}\,\mathrm{d}A_{z_n} = \frac{A_m A_n}{\left(4\pi R_0^2\right)^3} = \frac{1}{16\pi R_0^2}$$

and generally:

$$Y^{m,\,n}_{(p)l,\,k} = \frac{1}{2^{\,p+1}\pi R_0^2}$$

Then for the present case:

$$b_{ik}(\mathbf{x}_i,\mathbf{x}_k) = \frac{1}{4\pi R_0^2}\left(1+\rho+\rho^2+\cdots\right) = \frac{1}{4\pi R_0^2(1-\rho)} = \frac{1}{4\pi R_0^2\varepsilon}$$

If the series Eq. (13) is truncated at the Mth term, to obtain an accuracy better than 1%, one needs $M=2$ for $\rho=0.2$, $M=6$ for $\rho=0.5$ and $M=20$ for $\rho=0.8$.

From this very simple example, one can see that the solution reported by Eq. (13) is practicable only when surface reflectivities are small. However, in many practical situations, the peculiar form of Eq. (9c) allows an alternative solution of the problem.

Let us consider one of the surfaces (k) and suppose that it is possible to define a complete set of normalised orthogonal functions $\varphi^{(k)}_n(\mathbf{x}_k)$ such that:

$$\int_{A_k}\varphi^{(k)}_n(\mathbf{x}_k)\varphi^{(k)}_m(\mathbf{x}_k)\,\mathrm{d}A_k = \delta_{nm}$$

where δ_{nm} is the Kroeneker symbol. If this is possible for all the surfaces, then it is possible to expand any L_2-function defined over the surfaces A_l, A_k as:

$$a_{lk}(\mathbf{x}_l,\mathbf{x}_k) = \sum_{n,\,m=1}^{\infty}\alpha^{nm}_{lk}\varphi^{(l)}_n(\mathbf{x}_l)\varphi^{(k)}_m(\mathbf{x}_k) \tag{14a}$$

or

$$b_{lk}(\mathbf{x}_l,\mathbf{x}_k) = \sum_{n,\,m=1}^{\infty}\beta^{nm}_{lk}\varphi^{(l)}_n(\mathbf{x}_l)\varphi^{(k)}_m(\mathbf{x}_k) \tag{14b}$$

where the sets $\varphi^{(k)}_n(\mathbf{x}_k)$ and $\varphi^{(i)}_m(\mathbf{x}_i)$ can be different. Eq. (9c), which can be re-written as a system of integral equations:

$$b_{lk}(\mathbf{x},\mathbf{y}) = a_{lk}(\mathbf{x},\mathbf{y}) + \sum_{n=1}^{N}\rho_n\int_{A_n} a_{ln}(\mathbf{x}_l,\mathbf{z}_n)b_{nk}(\mathbf{z}_n,\mathbf{y}_k)\,\mathrm{d}A_{z_n}$$

can now be transformed in a set of linear algebraic equations by Eqs. (14a) and (14b):

$$\beta^{nm}_{lk} = \alpha^{nm}_{lk} + \sum_{j=1}^{N}\rho_j\alpha^{np}_{lj}\beta^{pm}_{jk} \tag{15}$$

From a practical point of view, the series in Eqs. (14a) and (14b) can be truncated at a certain term, the functions:

$$\hat{a}_{lk}(\mathbf{x}_l, \mathbf{x}_k) = \sum_{n,\, m=1}^{M} \alpha_{lk}^{nm} \varphi_n^{(l)}(\mathbf{x}_l) \varphi_m^{(k)}(\mathbf{x}_k) \tag{16}$$

are an approximation of a_{lk} and the "distance" $\|a_{lk} - \hat{a}_{lk}\| = \int_{A_l} \int_{A_k} (a_{lk} - \hat{a}_{lk})^2 \, \mathrm{d}A_l \, \mathrm{d}A_k$ decreases when M increases. After truncation, the solution of the finite set of Eq. (15) can be found by ordinary methods, obtaining the coefficients: $\hat{\beta}_{lk}^{nm}$ and the functions:

$$\hat{b}_{lk}(\mathbf{x}_l, \mathbf{x}_k) = \sum_{n,\, m=1}^{M} \hat{\beta}_{lk}^{nm} \varphi_n^{(l)}(\mathbf{x}_l) \varphi_m^{(k)}(\mathbf{x}_k) \tag{17}$$

that can be considered an approximation of the actual solution $b_{ik}(\mathbf{x}_i, \mathbf{x}_k)$; the difference: $\omega(\mathbf{x}_i, \mathbf{x}_k) = b_{ik}(\mathbf{x}_i, \mathbf{x}_k) - \hat{b}_{ik}(\mathbf{x}_i, \mathbf{x}_k)$ is equal to:

$$\omega(\mathbf{x}_i, \mathbf{x}_k) = \sum_{n,\, m=1}^{M} \left(\beta_{ik}^{nm} - \hat{\beta}_{ik}^{nm} \right) \varphi_n^{(i)}(\mathbf{x}_i) \varphi_m^{(k)}(\mathbf{x}_k) + \sum_{n,\, m=M+1}^{\infty} \beta_{ik}^{nm} \varphi_n^{(i)}(\mathbf{x}_i) \varphi_m^{(k)}(\mathbf{x}_k)$$

which may be used for an estimation of the error.

In the next section, an example of this procedure will be discussed.

4. The rectangular cavity

Let us consider a rectangular cavity as in Fig. 2, by introducing the co-ordinate system (ξ, η, ζ) sketched in Fig. 3 the functions a_{ik} are:

$$a_{kk} = 0 \quad \forall k$$

$$a_{12}(\xi_1, \eta_1; \xi_2, \eta_2) = \frac{4c^2}{\pi \left[4c^2 + (\xi_2 - \xi_1)^2 + (\eta_2 - \eta_1)^2 \right]^2} = a_{21}(\xi_2, \eta_2; \xi_1, \eta_1)$$

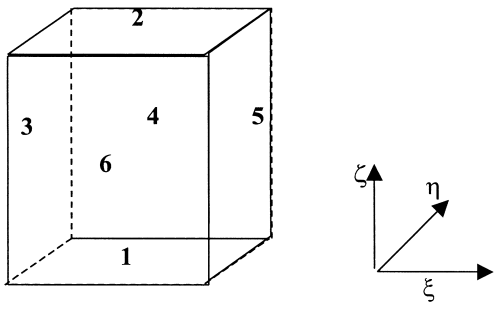

Fig. 2. Rectangular cavity and co-ordinate system.

$$a_{1k}(\xi_1, \eta_1; \zeta_k, \eta_k) = \frac{(\zeta_k + c)(\xi_1 + b)}{\pi \left[(\zeta_k + c)^2 + (\xi_1 + b)^2 + (\eta_k - \eta_1)^2 \right]^2} = a_{k1}(\zeta_k, \eta_k; \xi_1, \eta_1) \quad \text{for } k = 3, 4, 5, 6$$

$$a_{k2}(\zeta_k, \eta_k; \xi_2, \eta_2) = a_{k1}(\zeta_k, \eta_k; \xi_2, \eta_2) \quad \text{for } k = 3, 4, 5, 6$$

$$a_{2k}(\xi_2, \eta_2; \zeta_k, \eta_k) = a_{k2}(\zeta_k, \eta_k; \xi_2, \eta_2) \quad \text{for } k = 3, 4, 5, 6$$

$$a_{35}(\zeta_3, \eta_3; \zeta_5, \eta_5) = \frac{4b^2}{\pi \left[4b^2 + (\zeta_3 - \zeta_5)^2 + (\eta_3 - \eta_5)^2 \right]^2} = a_{53}(\zeta_5, \eta_5; \zeta_3, \eta_3)$$

$$a_{46}(\zeta_4, \eta_4; \zeta_6, \eta_6) = \frac{4a^2}{\pi \left[4a^2 + (\zeta_4 - \zeta_6)^2 + (\eta_4 - \eta_6)^2 \right]^2} = a_{64}(\zeta_6, \eta_6; \zeta_4, \eta_4)$$

$$a_{34}(\zeta_3, \eta_3; \zeta_4, \eta_4) = \frac{(\zeta_3 + c)(\zeta_4 + c)}{\pi \left[(\zeta_3 + c)^2 + (\zeta_4 + c)^2 + (\eta_3 - \eta_4)^2 \right]^2} = a_{43}(\zeta_4, \eta_4; \zeta_3, \eta_3)$$

$$a_{36}(\zeta_3, \eta_3; \zeta_6, \eta_6) = a_{63}(\zeta_6, \eta_6; \zeta_3, \eta_3) = a_{34}(\zeta_3, \eta_3; \zeta_6, \eta_6)$$

The set of orthogonal functions $\varphi_m^{(i)}(\mathbf{x}_i)$ chosen here is that of the normalised Legendre polynomials. A normalised Legendre polynomial of order n is defined as:

$$Z_n(x) = \frac{1}{n! 2^n} \sqrt{\frac{2n+1}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2 - 1)^n$$

where $x, y \in (-1, 1)$. For the present case, it is possible to set:

$$x_k = \frac{\xi_k}{b}; \quad y_k = \frac{\eta_k}{a}; \quad z_k = \frac{\zeta_k}{c};$$

for $k = 1, \ldots, 6$, and for each surface:

$$\varphi_{nm}^{(i)}(x_i, y_i) = Z_n(x_i) Z_m(y_i)$$

then, the functions a_{ik} can be approximated as:

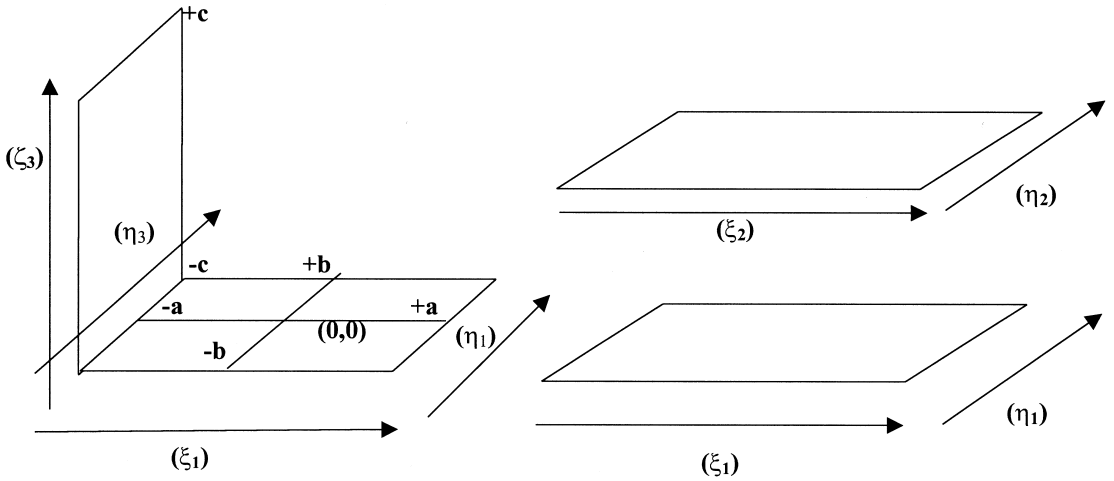

Fig. 3. Systems of co-ordinate used for couples of surfaces forming the rectangular cavity.

$$\hat{a}_{lk}(\mathbf{x}_l, \mathbf{x}_k) = \sum_{n,\,m,\,p,\,q=1}^{M} \alpha_{lk}^{nmpq} \varphi_{nm}^{(l)}(\mathbf{x}_l)\varphi_{pq}^{(k)}(\mathbf{x}_k)$$

where:

$$\alpha_{lk}^{nmpq} = \int_{A_l}\int_{A_k} a_{lk}(\mathbf{x}_l, \mathbf{x}_k)\varphi_{nm}^{(l)}(\mathbf{x}_l)\varphi_{pq}^{(k)}(\mathbf{x}_k)\,\mathrm{d}A_l\,\mathrm{d}A_k$$

The example of calculation shown here was performed by setting: $a = b = 1$, $c = 2$, and choosing $M = 3$ for the approximation, obtaining $\varepsilon < 1\%$ where:

$$\varepsilon = 100\frac{\|a_{lk} - \hat{a}_{lk}\|}{\|a_{lk}\|} = 100\frac{\int_{A_l}\int_{A_k}(a_{lk} - \hat{a}_{lk})^2\,\mathrm{d}A_l\,\mathrm{d}A_k}{\int_{A_l}\int_{A_k}(a_{lk})^2\,\mathrm{d}A_l\,\mathrm{d}A_k}$$

After the coefficients α_{lk}^{nmpq} were calculated (and Fig. 4 reports the absolute value of α_{12}^{nmpq} as a function of $N = n + m + p + q$), the solution b_{ik} can be found by

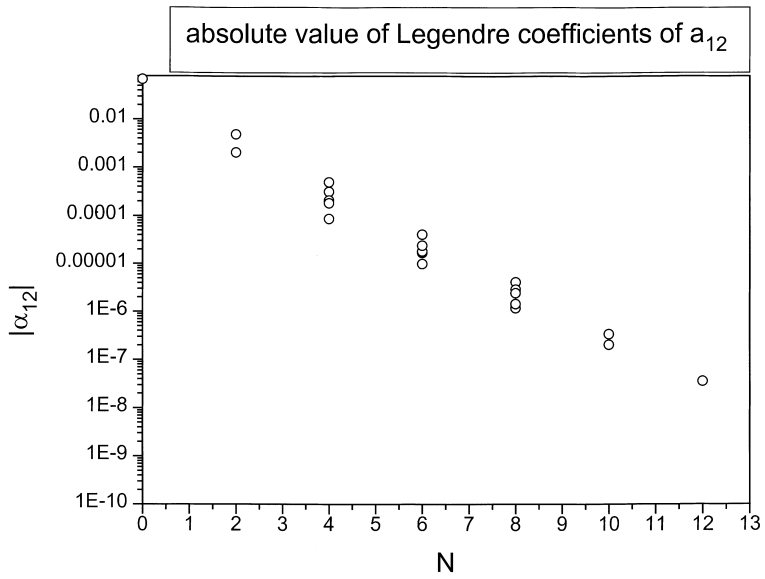

Fig. 4. Absolute value of Legendre coefficients of $a_{12}(\mathbf{x}, \mathbf{y})$ for the rectangular cavity as a function of the monomial order $N = n + m + p + q$.

solving the (finite) set of linear algebraic equations:

$$\hat{\beta}_{lk}^{nmpq} = \alpha_{lk}^{nmpq} + \sum_{j=1}^{N} \rho_j \sum_{s,\ t=0}^{M} \alpha_{lj}^{nmst} \hat{\beta}_{jk}^{stpq} \tag{15b}$$

by usual numerical routines.

An analytical solution, to which compare the numerical results, can be found by putting: $\rho_1 = \rho$, $\rho_k = 0$ for $k = 2, \ldots, 6$ (see Appendix B). In such case, Eq. (15b) simplifies:

$$\beta_{lk}^{nmpq} = \alpha_{lk}^{nmpq} + \rho \sum_{s,\ t=0}^{M} \alpha_{l1}^{nmst} \beta_{1k}^{stpq} \tag{15c}$$

and for the present case:

$$\beta_{11}^{nmpq} = 0 \quad \beta_{1k}^{nmpq} = \alpha_{1k}^{nmpq} \quad \beta_{k1}^{nmpq} = \alpha_{k1}^{nmpq}$$

$$\beta_{lk}^{nmpq} = \alpha_{lk}^{nmpq} + \rho \sum_{s,\ t=0}^{M} \alpha_{l1}^{nmst} \alpha_{1k}^{stpq} \quad \text{for } l, k \neq 1$$

from which it is possible to evaluate $b_{ik}(\mathbf{x}_i, \mathbf{x}_k)$. A comparison with the analytical solution is presented in Table 1 for the function $b_{22}(\mathbf{x}_2, \mathbf{x}_2')$ where the integral in:

$$b_{22}(\mathbf{x}_2, \mathbf{x}_2') = \rho \int_{A_1} a_{21}(\mathbf{x}_2, \mathbf{x}_1) a_{12}(\mathbf{x}_1, \mathbf{x}_2')\, \mathrm{d}A_1$$

was evaluated numerically.

As it can be seen, the accuracy $\varepsilon\% = \frac{2(\hat{b}_{22} - b_{22})}{\hat{b}_{22} + b_{22}} \cdot 100$ is better than 1% all over the domain of the function, showing that the method appears to be applicable at least when the cavity can be subdivided in an acceptably small number of surfaces.

Table 1
Values of the function $b_{22}(\mathbf{x}_2, \mathbf{x}_2')$: comparison between the analytical solution (b_{22}) and the approximated solution ($\hat{b}_{22}$) obtained by the method of the orthogonal functions for the rectangular cavity

ξ_2	η_2	ξ_2'	η_2'	$\hat{b}_{22}$	b_{22}	ε (%)
0	0	0	0	0.001345	1.35E−03	−0.61176
0	0	0	0.3	1.33E−03	1.34E−03	−0.43348
0	0	0	0.5	1.31E−03	1.32E−03	−0.33546
0	0	0	0.95	1.23E−03	1.23E−03	−0.34394
0	0	0.5	0	1.31E−03	1.32E−03	−0.33546
0	0	0.5	0.3	1.30E−03	1.30E−03	−0.16025
0	0	0.5	0.8	1.23E−03	1.23E−03	−7.08E−02
0	0	0.8	0	1.26E−03	1.26E−03	−0.3379
0	0	0.8	0.95	1.15E−03	1.15E−03	−0.07464
0	0.5	0	−0.8	1.22E−03	1.22E−03	−6.18E−02
0	0.5	0	−0.3	1.30E−03	1.30E−03	−0.15591
0	0.5	0.2	−0.7	1.24E−03	1.24E−03	5.03E−02
0	0.5	0.2	0	1.31E−03	1.31E−03	−0.30103
0	0.5	0.5	−0.7	1.21E−03	1.21E−03	0.283974
0	0.5	0.5	0	1.28E−03	1.28E−03	−5.92E−02
0	0.5	0.7	−0.95	1.13E−03	1.13E−03	0.270799
0	0.5	0.9	−0.3	1.19E−03	1.19E−03	0.160486
0	0.5	0.9	0	1.21E−03	1.21E−03	−1.23E−02
0.5	0.5	−0.9	0	1.17E−03	1.17E−03	0.253089
0.5	0.5	−0.9	0.5	1.15E−03	1.14E−03	0.517338
0.5	0.5	−0.9	−0.5	1.14E−03	1.13E−03	0.51847
0.5	0.5	−0.9	−0.9	1.07E−03	1.07E−03	0.566607
0.5	0.5	−0.7	0	1.21E−03	1.21E−03	0.286915
0.5	0.5	−0.7	0.5	1.19E−03	1.18E−03	0.553138
0.5	0.5	−0.7	0.9	1.12E−03	1.12E−03	0.603089
0.5	0.5	−0.7	−0.5	1.18E−03	1.17E−03	0.553476
0.5	0.5	−0.7	−0.9	1.11E−03	1.10E−03	0.592772
0.5	0.5	−0.5	0	1.24E−03	1.24E−03	0.211503
0.5	0.5	−0.5	0.5	1.22E−03	1.21E−03	0.480881
0.5	0.5	−0.5	−0.5	1.21E−03	1.20E−03	0.481239
0.5	0.5	−0.5	−0.9	1.14E−03	1.13E−03	0.51847
0.5	0.5	0	−0.9	1.17E−03	1.17E−03	0.253089
0.5	0.5	0	0.5	1.25E−03	1.25E−03	0.210158
0.5	0.5	0	0.9	1.19E−03	1.18E−03	0.261992
0.5	0.5	0.5	−0.5	1.22E−03	1.21E−03	0.480881
0.5	0.5	0.5	0.5	1.23E−03	1.22E−03	0.480526
0.5	0.5	0.5	0.9	1.16E−03	1.16E−03	0.527856
0.5	0.5	0.7	0	1.23E−03	1.22E−03	0.289449
0.5	0.5	0.9	0	1.19E−03	1.18E−03	0.261992
0.5	0.5	0.9	−0.5	1.15E−03	1.15E−03	0.528902
0.5	0.5	0.9	−0.9	1.09E−03	1.08E−03	0.575168

5. Conclusions

The proposed approach to the problem again leads to an integral Fredholm equation of second kind, then the mathematics of the problem is not simplified compared to the usual approaches, however, such equation contains an unique unknown function which depends only on the cavity geometry and surface properties. This separation of geometry and surface properties from surface temperature and heat flux allows to solve the integral equation once for a given geometry and surface properties and use the solution to compute heat fluxes from any surface temperature field (or vice versa) in a simple way.

The series solution of the equation can be useful when the surface reflectivity is relatively small, otherwise the number of term necessary for a given accuracy may become too large. A possible alternative method is based on the use of a truncated expansion by a complete set of ortho-normal function (like Legendre polynomials or others) which reduces the problem to the solution of a finite set of linear algebraic equations.

Appendix A

The equation $b(\mathbf{y},\ \mathbf{z}) - a(\mathbf{y},\ \mathbf{z}) = \int_A a(\mathbf{y},\ \mathbf{x})\rho(\mathbf{x})b(\mathbf{x},$

$\mathbf{z})\,\mathrm{d}A_x$ is a Fredholm integral equation of second kind, with kernel: $K(\mathbf{x},\ \mathbf{y}) = a(\mathbf{x},\ \mathbf{y})\rho(\mathbf{y})$. Its solution can be obtained under the form:

$$b(\mathbf{y},\mathbf{z}) = a(\mathbf{y},\mathbf{z}) + \int_A H(\mathbf{y},\mathbf{x})a(\mathbf{x},\mathbf{z})\,\mathrm{d}A_x \tag{A1}$$

where $H(\mathbf{x},\mathbf{y})$ is called "resolvent Kernel" and:

$$\begin{aligned} H(\mathbf{x},\mathbf{y}) &= K(\mathbf{x},\mathbf{y}) + K_2(\mathbf{x},\mathbf{y}) + K_3(\mathbf{x},\mathbf{y}) + \cdots \\ &= K(\mathbf{x},\mathbf{y}) + \sum_{n=2}^{\infty} K_n(\mathbf{x},\mathbf{y}) \end{aligned}$$

where:

$$K_2(\mathbf{x},\mathbf{y}) = \int_A K(\mathbf{x},\mathbf{z})K(\mathbf{z},\mathbf{y})\,\mathrm{d}A_z,$$

$$\begin{aligned} K_3(\mathbf{x},\mathbf{y}) &= \int_A\int_A K(\mathbf{x},\mathbf{z})K(\mathbf{z},\mathbf{t})K(\mathbf{t},\mathbf{y})\,\mathrm{d}A_t\,\mathrm{d}A_z \\ &= \int_A K(\mathbf{x},\mathbf{z})K_2(\mathbf{z},\mathbf{y})\,\mathrm{d}A \end{aligned}$$

and generally:

$$K_{n+1}(\mathbf{x},\mathbf{y}) = \int_A K(\mathbf{x},\mathbf{z})K_n(\mathbf{z},\mathbf{y})\,\mathrm{d}A_z.$$

The series converges provided that:

$$\int_A\int_A K^{\,2}(\mathbf{x},\mathbf{y})\,\mathrm{d}A_x\,\mathrm{d}A_y < 1 \text{ (see [10])}.$$

For the present case, the condition is: $\int_A\int_A a^2(\mathbf{x},\mathbf{y})\rho^2(\mathbf{y})\,\mathrm{d}A_x\,\mathrm{d}A_y < 1$ and, due to the fact that $\rho(\mathbf{y}) < 1$, the condition is satisfied if: $\int_A\int_A a^2(\mathbf{x},\ \mathbf{y})\,\mathrm{d}A_x\,\mathrm{d}A_y \le 1$.

Let us consider the integral: $\int_A\int_A a^2(\mathbf{x},\ \mathbf{y})\,\mathrm{d}A_x\,\mathrm{d}A_y$, by substituting the definition of $a(\mathbf{x},\mathbf{y})$:

$$\begin{aligned} &\int_A\int_A a^2(\mathbf{x},\mathbf{y})\,\mathrm{d}A_x\mathrm{d}A_y \\ &= \int_A\int_A \frac{\cos^2(\theta_x)\cos^2(\theta_y)}{\pi^2 R^4}\,\mathrm{d}A_x\,\mathrm{d}A_y \\ &= \int_A\int_A \frac{\cos(\theta_x)\cos(\theta_y)}{\pi^2}\,\mathrm{d}\omega_x\,\mathrm{d}\omega_y \\ &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\pi}\int_0^{\pi/2} \frac{\cos(\theta_x)\cos(\theta_y)}{\pi^2} \\ &\times \sin(\theta_y)\,\mathrm{d}\theta_y\,\mathrm{d}\varphi_y\,\sin(\theta_z)\,\mathrm{d}\theta_z\,\mathrm{d}\varphi_{z_y} \\ &= 4\int_0^{2\pi}\int_0^{\pi/2}\cos(\theta_y)\sin(\theta_y)\,\mathrm{d}\theta_y\,\mathrm{d}\varphi_y\int_0^{2\pi}\int_0^{\pi/2} \\ &\times \cos(\theta_z)\sin(\theta_z)\,\mathrm{d}\theta_z\,\mathrm{d}\varphi_z = 1 \end{aligned}$$

and the condition is satisfied.

Appendix B

A simple analytical solution of Eq. (9b) can be found under the following conditions: (a) the cavity can be decomposed into N different surfaces having constant properties (i.e. the reflectivity of each surface is independent of position); (b) only one surface (let say surface 1) has $0 < \rho < 1$ and it is convex (i.e. $a_{11} = 0$), for $k \neq 1$: $\rho_k = 0$.

Eq. (13) becomes:

$$b(\mathbf{y},\mathbf{z}) = a(\mathbf{y},\mathbf{z}) + \rho_1\int_{A_1} a(\mathbf{y},\mathbf{x})b(\mathbf{x},\mathbf{z})\,\mathrm{d}A_x$$

or, by using the functions $b_{ik}(\mathbf{x}_i,\mathbf{x}_k)$ and $a_{ik}(\mathbf{x}_i,\mathbf{x}_k)$:

$$b_{ik}(\mathbf{x}_i,\mathbf{x}_k) = a_{ik}(\mathbf{x}_i,\mathbf{x}_k) + \rho_1\int_{A_1} a_{i1}(\mathbf{x}_i,\mathbf{x}_1)b_{1k}(\mathbf{x}_1,\mathbf{x}_k)\,\mathrm{d}A_1$$

Then the following results are straightforward:

$$\begin{aligned} b_{11}(\mathbf{x}_1,\mathbf{x}_1) &= a_{11}(\mathbf{x}_1,\mathbf{x}_1) \\ &\quad + \rho_1\int_{A_1} a_{11}(\mathbf{x}_1,\mathbf{x}_1)b_{11}(\mathbf{x}_1,\mathbf{x}_k)\,\mathrm{d}A_1 = 0 \end{aligned}$$

$$\begin{aligned} b_{1k}(\mathbf{x}_1,\mathbf{x}_k) &= a_{1k}(\mathbf{x}_i,\mathbf{x}_k) + \rho_1\int_{A_1} a_{11}(\mathbf{x}_1,\mathbf{x}_1)b_{1k}(\mathbf{x}_1,\mathbf{x}_k)\,\mathrm{d}A_1 \\ &= a_{1k}(\mathbf{x}_1,\mathbf{x}_k) \quad \text{for } k\neq 1 \end{aligned}$$

$$\begin{aligned} b_{k1}(\mathbf{x}_k,\mathbf{x}_1) &= a_{k1}(\mathbf{x}_k,\mathbf{x}_1) + \rho_1\int_{A_1} a_{k1}(\mathbf{x}_k,\mathbf{x}_1)b_{11}(\mathbf{x}_1,\mathbf{x}_1)\,\mathrm{d}A_1 \\ &= a_{k1}(\mathbf{x}_k,\mathbf{x}_1) \quad \text{for } k\neq 1 \end{aligned}$$

$$b_{ik}(\mathbf{x}_i, \mathbf{x}_k) = a_{ik}(\mathbf{x}_i, \mathbf{x}_k) + \rho_1 \int_{A_1} a_{i1}(\mathbf{x}_i, \mathbf{x}_1) b_{1k}(\mathbf{x}_1, \mathbf{x}_k)\, \mathrm{d}A_1$$

$$= a_{ik}(\mathbf{x}_i, \mathbf{x}_k) + \rho_1 \int_{A_1} a_{i1}(\mathbf{x}_i, \mathbf{x}_1) a_{1k}(\mathbf{x}_1, \mathbf{x}_k)\, \mathrm{d}A_1$$

for $i, k \neq 1$;

which is the solution of the equation in a closed form.

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